

L7

# Locally presentable & Accessible categories

In the previous lecture we saw sketches, a categorical gadget useful to axiomatize stuff. So, that was a lecture about syntax.

This lecture will be about (their) semantics. We will introduce the notion of accessible categories.

Speaker

Syntax	Semantics
Mixed sketch	Accessible category
Limit / Colimit sketch	Locally presentable category

So, in a nutshell, accessible categories will be an extremely broad class of categories (anything axiomatizable by a sketch), while locally presentable ones will be a more restricted class, still including a vast majority of reasonable categories (of some kind of algebraic flavour).

We will focus on loc. pres. for time reason, the lecture has two main parts

Definition of accessible/pres. category

Representation theorem

First encounter with the small object argument. and orthogonality

# Part 1

Generators In the theory of accessible categories the notion of generator is quite important.

Def A generator  $\mathcal{G}$  (separator / separating set / generating set) in  $\mathcal{K}$  is a set of objects with the following property:

Given two <sup>different</sup> maps  $g, f: K \rightarrow K'$  there exist a map

$$g \xrightarrow{\exists} K \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{h} \end{array} K'$$

that does not make them equal.

Equivalently the functor

•  $\coprod_{g \in \mathcal{G}} K(g, -)$  is faithful.

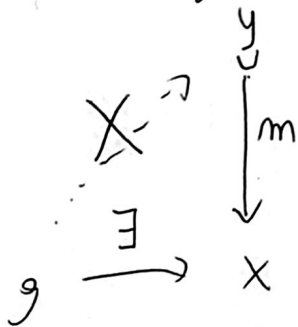
•  $N(i): K \rightarrow \text{Set}^{\mathcal{G}^{\text{op}}}$  is faithful

## Examples

Generator	Category
$\mathbb{Z}$	<del>All</del>
$\mathbb{R}$	$\text{Mod}(\mathbb{R})$
•	$\text{Set}$
•	$\text{Set}_{>0}$
$\mathbb{Z}_p$	$\text{Fid}$
•	$\text{Top}$

Previously in the course we mentioned the notion of dense generator. That is clearly a stronger notion. Not all generators are the same (!)

Def A generator is strong if it distinguishes isomorphism classes. I.e. given a proper subobject



there is a ~~map~~ that does not factor through the subobject.

Equivalently

- $\coprod_{g \in \mathcal{G}} K(g, -)$  is faithful + conservative
- $N(i)$  is faithful + conservative

Ex If  $K$  has equalizers a right adjoint is faithful + conservative iff it is conservative.

Rem All the ~~new~~ examples are strong but top.

Recap . Implications of generators.

Dense  $\Rightarrow$  strong  $\Rightarrow \neq$

Rem It is much easier to show that something is a generator. Density is a very hard thing to check.

Rem (Isbell)  $\mathcal{Y} \xrightarrow{i} \mathcal{K}$  is dense iff  $N(i)$  is fully faithful

dense	fully faithful
strong	conservative + faithful
$\neq$	faithful.

Small objects | In a category some objects have an "abstract cardinality", which is called "presentability rank". we already saw the notion of  $\aleph_1$ -ness.

Def An object is 1-presentable if its hom functor  $K(y, -)$  preserve  $\aleph_1$ -directed colimits.

Prop In Set,  $X$  is 1-present iff  $\text{card}(X) < \aleph_1$ .

Example Similar for groups and algebraic stuff

Ex In top? ~~only the small topological groups are 1-present~~

We have already met this notion in the lecture about varieties. Do you remember?

Prop  $\lambda$ -pres objects are closed under  $\lambda$ -small colimits.

## Accessibility

Def A category is  $\lambda$ -accessible if it has  $\lambda$ -directed colimits and a dense generator made of  $\lambda$ -presentable objects.

It is locally  $\lambda$ -presentable if it is  $\lambda$ -accessible and cocomplete.

Examples Set, ok. Set<sup>cop</sup>, ok.

Grp? Mod(R)? Top? It's hard to say - How do I test if the generator is dense?!

Thm A category is  $\lambda$ -accessible iff it has  $\lambda$ -dir colimits and a ~~set~~ set of  $\lambda$ -pres objects that generate under  $\lambda$ -directed colimits.

Rem ok, this is much better. Grp, Mod R are easy to check. Top? Colimits in Top are less trivial...

Thm A category is loc  $\lambda$ -presentable iff it is cocomplete and has a strong generator made of  $\lambda$ -pres objects.

Rem Top is not.

## Part 2 "Representation theorem"

Thm Consider the inclusion  $\text{Pres}(K) \xrightarrow{i} K$ , where  $K$  is  $\lambda$ -accessible. Then the  $\lambda$ -functor

$$N(i): K \longrightarrow \text{Psh}(\text{Pres}_\lambda(K))$$

is fully faithful & preserves  $\lambda$ -directed colimits.

Question What is the image? We will only answer in the presentable case.

Thm If  $K$  is loc pres then  $N(i)$  has a left adjoint. So it is reflective in  $\text{Psh}(\text{Pres}_\lambda(K))$ .  
Moreover  $N(i)$  lands in  $\text{Lex}(\text{Pres}_\lambda(K)^{\text{op}}, \text{Set})$ .

$$\begin{array}{ccc} K & \longrightarrow & \text{Psh}(\text{Pres}_\lambda(K)) \\ & \searrow & \nearrow \\ & & \text{Cont}_\lambda(\text{Pres}_\lambda(K)^{\text{op}}, \text{Set}) \end{array}$$

Thm

$$K \simeq \text{Cont}_\lambda(\text{Pres}_\lambda(K)^{\text{op}}, \text{Set}) \text{ for all locally } \lambda\text{-pres categories}$$

Ah... now I see some limit sketch entering the picture...

For the rest of the lecture we follow  
closely LPAC 1.C. pag 27-36.

Rem 1.33(7) & (8) are very important  
but there is a bad typo in the book !!