L7 Locally presentable & Accessible categories

In the previous lecture we saw <u>sketches</u>, a categorical gadget useful to axiomatize stuff. So, that was a lecture about <u>syntax</u>.
This lecture will be about (their) semantics. We will introduce the notion of <u>accessible categories</u>.

Spoiler

	Syntax	Semantics
	Mixed sketch	Accessible category
	Limit/Colimit sketch	Locally presentable category

So, in a nutshell, <u>accessible categories</u> will be an extremely broad class of categories (anything axiomatizable by a sketch), while <u>locally presentable</u> ~~ones~~ ones will be a more restricted class, still including a vast majority of reasonable categories (of some kind of algebraic flavour).

We will focus on loc. pres for time reason, the lecture has two main parts

Definition of accessible/pres category	Representation theorem First encounter with the <u>small object argument</u> and <u>orthogonality</u>

Part 1

$\boxed{\text{Generators}}$ In the theory of accessible categories the notion of generator is quite important.

Def A generator $\mathcal{G}$ (separator / separating set / generating set) in $\mathcal{K}$ is a set of objects with the following property:

Given two different maps $h, f : k \longrightarrow k'$ there exist a map

$$\begin{tikzcd} g \arrow[r, "\exists"] & k \arrow[r, shift left, "f"] \arrow[r, shift right, "h"'] & k' \end{tikzcd}$$

that does not make them equal.

Equivalently the functor

- $\coprod_{g \in \mathcal{G}} \mathcal{K}(g, -)$ is faithful.
- $N(i) : \mathcal{K} \longrightarrow Set^{\mathcal{G}^{op}}$ is faithful

Examples

Generator	Category
$\mathbb{Z}$	Ab
R	Mod(R)
$\bullet$	Set
$\bullet$	$Set_{>0}$
$\mathbb{Z}_p$, $\mathbb{Q}$	Fld
$\bullet$	Top

Previously in the course we mentioned the notion of <u>dense</u> generator. That is clearly a stronger notion. Not all generators are the same (!)

<u>Def</u> A generator is <u>strong</u> if it distinguishes isomorphism classes. I.e. given a proper subobject

$$\begin{tikzcd} & Y \arrow[d, hook, "m"] \\ g \arrow[ur, dashed] \arrow[r, "\exists"] & X \end{tikzcd}$$

there is a map that does not factor through the subobject.

Equivalently

- $\coprod_{g \in \mathcal{G}} K(g, -)$ is faithful + conservative
- $N(i)$ is faithful + conservative

<u>Ex</u> If K has equalizers a right adjoint is faithful + conservative iff it is conservative.

<u>Rem</u> All the previous examples are strong $\boxed{\text{but Top}}$.

Recap . Implications of generators.

Dense $\Rightarrow$ strong $\Rightarrow$ $\emptyset$

Rem It is somuch easier to show that something is a generator. Density is a very hard thing to check.

Rem (Isbell) $\mathcal{G} \xhookrightarrow{i} \mathcal{K}$ is dense iff $N(i)$ is fully faithful

dense	fully faithful
strong	conservative + faithful
$\emptyset$	faithful.

Small objects In a category some objects have an "abstract cardinality", which is called "presentability rank". We already saw the notion of Tiny-ness.

Def An object is λ-presentable if its hom functor $\mathcal{K}(g,-)$ preserve λ-directed colimits.

Prop In Set, X is λ-pres iff $\mathrm{card}(X) < \lambda$.

Example Similar for groups and algebraic stuff

Ex In top? ~~[illegible]~~

We have already met this notion in the lecture about varieties. Do you remember?

Prop λ-pres objects are closed under λ-small colimits.

Accessibility

Def A category is λ-accessible if it has λ-directed colimits and a dense generator made of λ-presentable objects.
It is locally λ-presentable if it is λ-accessible and cocomplete.

Examples Set, ok. $Set^{C^{op}}$, ok.

Grp ? Mod(R) ? Top ? It's hard to say.
How do I test if the generator is dense ?!

thm A category is λ-accessible IFF it has λ-dir colimits and a set of λ-pres objects that generates under λ-directed colimits.

Rem ok, this is much better. Grp, Mod R are easy to check. Top ? Colimits in Top are less trivial...

thm A category is loc λ-presentable IFF it is cocomplete and has a strong generator made of λ-pres objects.

Rem Top is not.

Part 2 "Representation theorem"

<u>thm</u> Consider the inclusion $\mathrm{Pres}_\lambda(K) \xhookrightarrow{i} K$, where K is λ-accessible. Then the functor

$$N(i) : K \longrightarrow \mathrm{Psh}(\mathrm{Pres}_\lambda(K))$$

is fully faithful & preserve λ-directed colimits.

<u>Question</u> What is the image? We will only answer in the presentable case.

<u>thm</u> If K is loc pres then $N(i)$ has a left adjoint. So it is reflective in $\mathrm{Psh}(\mathrm{Pres}_\lambda(K))$. Moreover $N(i)$ lands in $\mathrm{Lex}(\mathrm{Pres}_\lambda(K)^{op}, \mathrm{Set})$.

$$\begin{tikzcd}
K \arrow[rr] \arrow[dr] & & \mathrm{Psh}(\mathrm{Pres}_\lambda(K)) \\
& \mathrm{Cont}_\lambda(\mathrm{Pres}_\lambda(K)^{op}, \mathrm{Set}) \arrow[ur] &
\end{tikzcd}$$

<u>thm</u>

$K \simeq \mathrm{Cont}_\lambda(\mathrm{Pres}_\lambda(K)^{op}, \mathrm{Set})$ for all locally λ-pres category.

Ah... now I see some limit sketch entering the picture...

For the rest of the lecture we follow closely LPAC 1.C. pag 27 – 36.

Rem 1.33 (7) & (8) are very important but there is a bad typo in the book !!